COLLISION WITH AN OBSTACLE OF A BODY CONTAINING A VISCOUS FLUID

PMM Vol. 33, №5, 1969, pp. 899-904

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(Received December 27, 1968)

The dynamics of a body with cavities completely or partially filled with fluid (in most cases an ideal fluid) is the subject of many papers, the first of which was by Zhukovskii [1]. Surveys of the principal approaches and results will be found in monographs [2, 3]. In dealing with the collision problem the fluids are usually assumed to be ideal. This assumption is valid for fluids which must be considered viscous under ordinary conditions, since very large forces develop during collision. On the other hand, if the obstacle with which the body collides has a finite elasticity, then allowance for viscosity is necessary. The present paper concerns the effect of viscosity on the interaction of a body with an obstacle. Our formulation here is the same as that used in [4]. For simplicity we shall consider the case of motion of a body which can be described by means of a single generalized coordinate.

1. Let a rigid body containing a cavity Ω of arbitrary shape completely filled with a viscous incompressible fluid rotate about a fixed axis and collide with an obstacle at the instant $t = 0$. If $M(\varphi, t)$ is the moment of the forces exerted by the obstacle on the body, J the moment of inertia of the body, φ the angle of rotation and $N(t)$ the moment of the forces exerted on the body by the fluid, then the equation of motion of the body in the period $0 < t < \Delta t$ during which the body is in contact with the obstacle is of the form

$$J\frac{d^2\varphi}{dt^2} - M(\varphi, t) = N(t) \tag{1.1}$$

In the case of an absolutely elastic obstacle we can set $M(\varphi, t) = -k\varphi$, where $k = \text{const}$.

We know (e.g. see [2]) that the motion of a viscous fluid is described by the following system of equations:

$$\frac{\partial u}{\partial t} + u\nabla u = X - \frac{1}{\rho}\operatorname{grad} p + \nu\nabla^2 u, \qquad \operatorname{div} u = 0 \tag{1.2}$$

Here u is the velocity, ρ, ν, p the density, viscosity, and pressure of the fluid, respectively, and X the external body forces. Since the collision occurs within a small time interval Δt, since the displacements of the body and fluid particles in this interval are small, and since the forces acting are large, we can neglect the ordinary forces X and set $(u\nabla)u = 0$ (equality of the convective terms to zero [5]). To solve the problem of motion of the body-fluid system we must solve Eqs. (1.1), (1.2) simultaneously under the no-slip conditions (the velocity of the fluid at the boundary of the cavity is equal to that of the points of the boundary) and a certain initial condition (for a given fluid velocity field and a given velocity of the body at $t = 0$).

In the case of a linear function $M(\varphi, t)$ we can use the Duhamel principle; to find $M(t)$ under zero initial conditions we only need to know the solution of the problem of motion of the fluid initially at rest upon instantaneous acceleration of the body to the constant angular velocity $d\varphi / dt$ [4] (Problem 1). Let $L(t)$ be the moment of forces exerted on the body by the fluid. Then

$$N(t) = \int_0^t \frac{d^2\varphi(\tau)}{d\tau^2} L(t-\tau)\, d\tau \tag{1.3}$$

The problem with zero initial conditions therefore reduces to the solution of integro-differential equation (1.1). Examples of its solution can be found in [4, 6]. To solve the general problem we must also solve the hydrodynamic problem of decay of the specified velocity field as the body comes to rest (Problem 2), calculate the corresponding moment of forces $L_0(t)$, and add it to $L(t)$ in (1.3).

Let a body rotating at some velocity ω and containing a fluid with a given velocity field at $t = 0$ be brought to rest instantaneously. Since $\omega > 0$ and $\nu < \infty$, there exists a small t_0 such that the fluid behaves as an ideal fluid for all t from the interval $0 < t < t_0$. The resulting velocity field decays in the subsequent instants. This behavior of the solutions is due to the fact that the equations of motion of the fluid combine certain properties of elliptic and parabolic equations. In the case of an elastic collision, the character of the latter depends essentially on the relationship between the elasticity and viscosity coefficients. For very large coefficients of elasticity the instant of detachment of the body from the obstacle ($\Delta t < t_0$) and the effective moment of momentum of the fluid after detachment are smaller than in the case of a solidified fluid [1] and energy is lost through dissipation. For smaller values of the coefficient of elasticity the viscosity is important during the actual collision; for $\Delta t \gg t_0$ the fluid behaves almost as a solid, and energy dissipation is slight (as in the case considered in [4]).

The fluid pressure p which must be known in order to compute $N(t)$ must be determined from formula (1.2) (provided the velocity field is known). In the simplest case of collision of a nonrotating body there is no relative motion of the fluid [1] and the viscosity, provided it is finite, is of no significance. Equation (1.2) then degenerates into

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\operatorname{grad} p$$

from which we see that $\operatorname{grad} p = f(t)$ and the pressure distribution is purely gyrostatic. The function $f(t)$ must be determined by solving the problem of the interaction of an absolutely rigid body with an elastic obstacle.

2. In this section we consider the decay of the fluid velocity field upon sudden coming to rest of the body, i.e. the case of interaction with an ideally inelastic obstacle (Problem 2) and describe a procedure for solving the problem. We note that Problem 1 is reducible to a special case of Problem 2. In subsequent sections we shall consider specific examples and results. For simplicity, let us consider the two-dimensional case (the axisymmetric case can be analyzed in completely analogous fashion). Making use of (1.2), we set

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x} \tag{2.1}$$

Substituting these expressions into (1.2) and recalling the simplifications of this equation noted in Sect. 1, we obtain

$$\frac{\partial\nabla^2\psi}{\partial t} = \nu\nabla^4\psi \tag{2.2}$$

Since the body is at rest for $t > 0$, it follows that the boundary and initial conditions

$$\psi|_S = \frac{\partial\psi}{\partial n}\Big|_S = 0, \qquad \psi|_{t=0} = \psi_0(x, y) \tag{2.3}$$

must be fulfilled.

Here n is the direction of the normal to the boundary S of the cavity; $\psi_0(x, y)$ is a given function. It is important to note that ψ_0 is not arbitrary, but describes the velocity field which arises in the fluid after the body has come to rest. It can be found by solving the problem of motion of an ideal fluid, which reduces to the solution of the Neumann problem.

The solution of problem (2.2), (2.3) is obtainable in the form of a Fourier series,

$$\psi = \sum_{k=1}^{\infty} c_k e^{-\nu\lambda_k t} \chi_k(x, y) \tag{2.4}$$

where χ_k are the eigenfunctions of the equation

$$\nabla^4\chi_k + \lambda_k \nabla^2\chi_k = 0 \tag{2.5}$$

Problems of this type occur in the theroy of elasticity [7]. Thus, we already have proofs of the countability of the λ_k, of their positiveness, and of the completeness of the system of functions χ_k; we also have methods for finding the λ_k and χ_k. For example, we can use the Ritz method to solve the above problem. The conditions of solvability of problem (2.2), (2.3) are also considered in [8]. Analytical methods are applicable only in those cases where the trajectories of the fluid particles remain constant in space, so that (2.5) reduces to a second-order equation (e.g. see [6]). Having found the λ_k and χ_k, we need merely use (2.4) to find the c_k as the coefficients of the Fourier expansion of the function ψ_0.

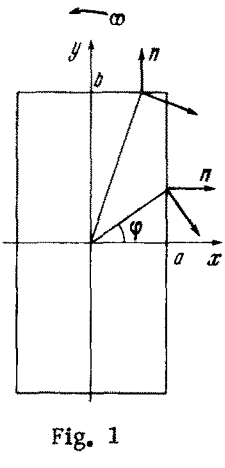

Fig. 1

3. Let the cavity be rectangular with a length/width ratio b/a (Fig. 1) and let the fluid rotate together with the body at the constant angular velocity ω prior to collision (this is the simplest and most important case). The body comes to a stop at the instant $t = 0$. In attempting to find $\psi_0(x, y)$ it is convenient to consider the motion of the fluid in a coordinate system rotating at the angular velocity ω. The relative equations of motion are of the form [5]

$$\frac{\partial u'}{\partial t} - 2\omega v' = -\frac{1}{\rho}\frac{\partial p'}{\partial x}, \qquad \frac{\partial v'}{\partial t} + 2\omega u' = -\frac{1}{\rho}\frac{\partial p'}{\partial y} \tag{3.1}$$

where the primed letters denote the values of the corresponding quantities in the relative coordinate system. The continuity equation remains unchanged in form. Eliminating p' and introducing ψ' by means of formulas analogous to (2.1), we obtain an equation for ψ'

$$\nabla^2\psi' = 0 \tag{3.2}$$

The coming to rest of the body at $t = 0$ is equivalent in the relative coordinate system to the body being instantaneously set in motion from the rest state at the angular velocity $-\omega$. The boundary conditions (Fig. 1) are therefore as follows:

$$-\frac{\partial\psi'}{\partial y}\bigg|_{x=a} = -\frac{\partial\psi'}{\partial y}\bigg|_{x=-a} = \omega y, \qquad \frac{\partial\psi'}{\partial x}\bigg|_{y=b} = \frac{\partial\psi'}{\partial x}\bigg|_{y=-b} = -\omega x \tag{3.3}$$

Integration over the contour S reduces problem (3.2), (3.3) to the Dirichlet problem. In fact, if we assume that $\psi'(a, b) = 0$, the appropriate computations yield

$$\psi'(a, y) = \psi'(-a, y) = -\tfrac{1}{2}\omega\,(y^2 - b^2)$$
$$\psi'(x, b) = \psi'(x, -b) = -\tfrac{1}{2}\omega\,(x^2 - a^2) \tag{3.4}$$

We can attempt to find the solution of problem (3.2), (3.4) in the form

$$\psi'(x,y)=\sum_{n=1}^{\infty}\left[A_n \operatorname{ch}\frac{(2n-1)\pi}{2a}y\cos\frac{(2n-1)\pi}{2a}x+B_n\cos\frac{(2n-1)\pi}{2b}y\operatorname{ch}\frac{(2n-1)\pi}{2b}x\right] \tag{3.5}$$

Substituting (3.5) into (3.4) and integrating, we obtain the following expressions for A_n and B_n:

$$A_n=\frac{16\omega a^2(-1)^{n-1}}{(2n-1)^3\pi^3}\operatorname{sch}\frac{(2n-1)\pi b}{2a},\qquad B_n=\frac{16\omega b^2(-1)^{n-1}}{(2n-1)^3\pi^3}\operatorname{sch}\frac{(2n-1)\pi a}{2b}$$

Since the absolute velocity of the fluid is equal to the sum of the relative and translational velocities, we can write

$$\psi_0(x,y)=\psi_1+\psi',\quad \psi_1(x,y)={}^1/_2\omega(x^2+y^2) \tag{3.6}$$

Here $\psi_1(x,y)$ is the function describing rotation at the angular velocity ω. Thus, the first part of Problem 2 has been solved.

We can find the λ_k and χ_k by the Ritz method. Since the function $\psi_0(x,y)$ is symmetric, we can find an approximate expression for χ in the form [7]

$$x\approx(x^2-a^2)^2(y^2-b^2)^2(a_1+a_2x^2+a_3y^2) \tag{3.7}$$

where a_k are unknown coefficients. Determination of the eigenvalues reduces to finding the minimum of the functional [7]

$$\int_\Omega(\nabla^2\chi)^2\,d\Omega$$

under conditions (2.3) and the additional condition

$$\|\chi\|^2=\int_\Omega \operatorname{grad}^2\chi\,d\Omega=1 \tag{3.8}$$

This brings us to the determination of the values of λ_k for which the system

$$\sum_{k=1}^{3}a_k[(\nabla^2\varphi_k,\nabla^2\varphi_m)-\lambda(\nabla\varphi_k,\nabla\varphi_m)]=0\qquad(m=1,2,3) \tag{3.9}$$

has a nontrivial solution. Here the parentheses enclose the scalar products of the corresponding functions, i.e. the integrals of their products over the domain Ω. The equality to zero of the determinant of system (3.9) implies a cubic equation in λ. Its solution yields the following values for the first eigenvalue: $\lambda_1=2.30/b^2$ for $a=b$, $\lambda_1=9.50/b^2$ for $a=2b$ and $\lambda_1\to10.1/b^2$ for $a\to\infty$.

The system of functions χ_k is complete in the space with norm (3.8), and the familiar theorems [7] imply its completeness in the sense of convergence in the mean. Let $a=2b$. Substitution of λ_1 into system (3.9) yields an approximate formula for χ_1,

$$\chi_1=b^{-11}(x^2-a^2)^2(y^2-b^2)^2(0.017b^2-0.025x^2-y^2) \tag{3.10}$$

In the last stage of decay, when the first term in (2.4) is dominant, the solution (as can be shown by computing the first Fourier coefficient) is of the form

$$\psi=-0.58\omega b^3\chi_1(x,y)\exp(-9.5\nu t/b^2)$$

4. As a further example let us consider the following problem. A plane of mass m per unit area supervened by a layer of viscous fluid of thickness h which initially moves at the same velocity as the plane collides with an elastic obstacle at the instant $t=0$; the force acting per unit area of the plane is $F(t)$. We are to determine the motion of the plane after the collision. This example can serve as a model of the problem of collision of a rotating body containing a fluid in an annular gap.

Equation (1.1) in this case becomes

$$m\frac{d^2x}{dt^2} - F(t) = \int_0^t \frac{d^2x(\tau)}{d\tau^2}\sigma(t-\tau)\,d\tau \qquad (4.1)$$

where the x-axis lies along the direction of motion. To determine $\sigma(t)$ we must solve the hydrodynamic problem of finding the fluid velocity field when the plane is instantaneously set in motion at a constant velocity, i.e. we must find the function $u(y, t)$ satisfying the equation

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (4.2)$$

under the boundary and initial conditions

$$\frac{\partial u(h,t)}{\partial y} = 0\ (t \geqslant 0), \quad u(0,t) = 1\ (t > 0), \quad u(y,0) = 0 \qquad (4.3)$$

Then $\sigma(t) = \mu\,\partial u/\partial y\,|_{y=0}$. Solution of the hydrodynamic problem in this case yields second-order equation (4.2) instead of (2.2), since continuity equation (1.3) is satisfied trivially. If $u(y, t)$ is the solution of the formulated problem, then $v = 1 - u(y, t)$ represents the solution of the problem of decay of the velocity field under the conditions

$$\frac{\partial v(h,t)}{\partial y} = 0\ (t \geqslant 0), \quad v(0,t) = 0\ (t > 0), \quad v(y,0) = 1 \qquad (4.4)$$

The solution of the latter problem can also be sought in the form (2.4), where χ_k are the eigenfunctions of the equation $\partial^2\chi_k / \partial y^2 + \lambda_k\chi_k = 0$

under conditions (4.4). The computations yield the result

$$\lambda_k = (2k-1)^2\pi^2 / 4h^2, \quad \chi_k = \sin[(2k-1)\pi y / 2h]$$

To determine c_k from (2.4) we must solve the problem of expanding function (4.4) in a Fourier series in the functions $\chi_k(y)$. The appropriate computations yield $c_k = 4 / (2k-1)\pi$. Series (2.4) converges rapidly for $t > 0$; for $t = 0$ it converges rapidly in every closed domain lying entirely inside Ω. At the boundary between the fluid and the wall we have the formula

$$\sigma(t) = \frac{2\mu}{h}\sum_{k=1}^{\infty}\exp(-\lambda_k\nu t)$$

Thus, an infinite force per unit area acts at the initial instant. This singularity is integrable, however.

The case of high viscosity, i.e. the case where the fluid moves almost as a solid body, was investigated in [4]. Let us now consider the other extreme case where the viscosity of the fluid during collision is negligible, i.e. where $F(t) = mU\delta(t)$. We assume for simplicity that the plane and fluid were at rest prior to collision. The general case was obtained by adding the translational displacement of the entire system to the flow under consideration; the rate of this displacement depends on the character of the collision (the properties of the obstacle). Substituting the function $F(t)$ into (4.1) and dividing the result by $m\nu^2 / h^3$, we can express the equation of motion of the plane in dimensionless form,

$$\frac{d^2x_1}{dt_1^2} - U_1 2\frac{\rho h}{m}\sigma_1(t_1) = 2\frac{\rho h}{m}\int_0^{t_1}\frac{d^2x_1(\tau)}{d\tau^2}\sigma_1(t_1-\tau)\,d\tau, \quad \sigma_1 = \sum_{k=1}^{\infty}\exp(-\varkappa_k t) \qquad (4.5)$$

$$x_1 = x / h,\ t_1 = t\nu / h^2,\ U_1 = Uh / \nu,\ \sigma_1 = \sigma h^2 / m\nu,\ \varkappa_k = \lambda_k h^2$$

Let us subject (4.5) to Laplace transformation.

$$X(p)=\int_0^\infty x_1(t_1)\,e^{-pt_1}dt_1,\quad x(0)=0,\quad x'(0)=U$$

Changing the order of integration in the right side yields the relation

$$X(p)=U_1\left\{p^2\left[1+2\,\frac{\rho h}{m}\sum_{k=1}^{\infty}(p+\varkappa_k)^{-1}\right]\right\}^{-1} \tag{4.6}$$

The function $X(p)$ has a second-order pole at $p=0$ and a countable set of first-order poles on the negative part of the real axis. Hence, as we can see from the inverse Laplace transformation formula, the solution $x_1(t_1)$ is of the form

$$x_1(t_1)=Vt_1+\sum_{k=1}^{\infty}a_k\exp(-\gamma_k t)+W \tag{4.7}$$

It is easy to see that $W=-(a_1+a_2+a_3+\ldots)$ and that $\gamma_k>0$. All of the constants in (4.7) can be determined from (4.6). For example, by computing the residue at the origin we obtain the formula $V=U_1(1+\rho h/m)^{-1}$ which is readily obtainable from the conditions of conservation of momentum.

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Translated by A. Y.